16/11/23 MATH4030 Tutorial.

Announcements:

- HW5 due 27/11

Recall: - a geometric quantity is called "intrinsic" if it only depends on the metric [g]
in other words, if it is invariant under isometries.

- a diffeomorphism $\varphi: M \to N$ is an isometry if $\forall p \in M$, $d\varphi_p : T_pM \to T_{\varphi(p)}N$
preserves the metric, ie.

$$g_M(V, W)_p = g_N(d\varphi_p(V), d\varphi_p(W))_{\varphi(p)}.$$

$$\text{"} \langle V, W \rangle_p = \langle d\varphi_p(V), d\varphi_p(W) \rangle_{\varphi(p)} \text{"}.$$

- a diffeomorphism $\varphi: M \to N$ is conformal if $\forall p \in M$, $V, W \in T_pM$,

$$g_N(d\varphi_p(V), d\varphi_p(W))_{\varphi(p)} = \lambda^2 g_M(V, W)_p$$

for a nowhere zero smooth fn λ on M.

isometry $\iff$ conformal and $\lambda \equiv 1$.

Q1: A diffeomorphism is area preserving if $A(R) = A(\varphi(R))$ for any region $R \subset M$.

Show that if φ is area preserving and conformal, then φ is an isometry.

Hint: Use the fact that if $X(u,v)$ param. M

$\bar{X}(u,v)$ param N $(\bar{X} = \varphi \circ X)$.

then $d\varphi_p(X_u) = \bar{X}_u$

$d\varphi_p(X_v) = \bar{X}_v$.

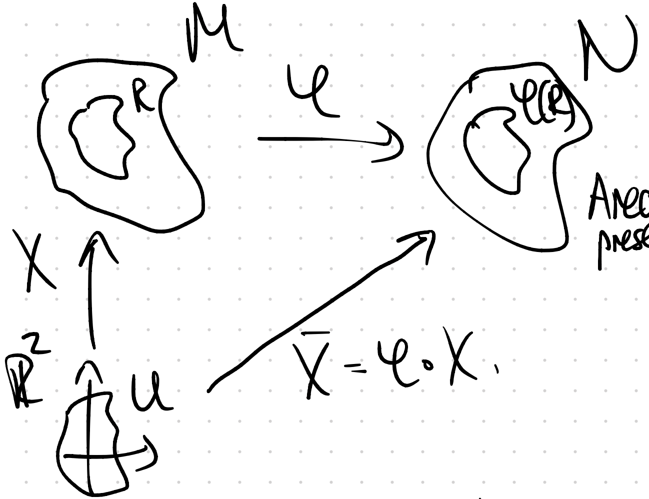
Pf: Recall $A(R) = \int_R \sqrt{EG-F^2} = \int_{U \subset \mathbb{R}^2} \sqrt{EG-F^2}\, du\, dv$ where $X(U) = R$.

Using the fact above, we have

$\bar{E} = \langle \bar{X}_u, \bar{X}_u \rangle = \langle d\varphi_p(X_u), d\varphi_p(X_u) \rangle = \lambda^2 \langle X_u, X_u \rangle = \lambda^2 E$.

$\bar{F} = \lambda^2 F$, $\bar{G} = \lambda^2 G$.

So we have that $\sqrt{\bar{E}\bar{G}-\bar{F}^2} = \sqrt{\lambda^4(EG-F^2)} = \lambda^2\sqrt{EG-F^2}$.



Area preserving $\to \varphi$

$$A(R) = \int_U \sqrt{EG-F^2}\,du\,dv.$$

$$A(\varphi(R)) = \int_U \sqrt{\bar{E}\bar{G}-\bar{F}^2}\,du\,dv \underset{\text{conformal}}{=} \int_U \lambda^2\sqrt{EG-F^2}\,du\,dv.$$

$\Rightarrow \lambda \equiv 1$. so φ is an isometry ✓.

Recall: $X_{ij} = \partial_i \partial_j X$ (as a vector in $\mathbb{R}^3$) $\quad X = X(u,v) = X(u^1, u^2)$.

$$= \Gamma_{ij}^{k} X_k + h_{ij} N. \quad (\text{b/c } \mathbb{R}^3 = \operatorname{span}\{X_1, X_2, N\}.)$$

Einstein summation notation

$$= \sum_{k=1}^{2} \Gamma_{ij}^{k} X_k + h_{ij} N,$$

Lemma (Lecture):

- $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$
- $\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right). \quad (1)$

$$= \frac{1}{2} \sum_{l=1}^{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right).$$

Q2: Compute the Christoffel symbols using (1) above of a surface of revolution given by

$$X(u^1, u^2) = (f(u^2)\cos u^1, f(u^2)\sin u^1, g(u^2)).$$

解: $E = f(u^2)^2$, $F = 0$, $G = (f'(u^2))^2 + (g'(u^2))^2$.

$$g = \begin{bmatrix} f(u^2)^2 & 0 \\ 0 & f'(u^2)^2 + g'(u^2)^2 \end{bmatrix}, \quad g^{-1} = \frac{1}{f^2((f')^2 + (g')^2)} \begin{bmatrix} (f')^2 + (g')^2 & 0 \\ 0 & f^2 \end{bmatrix}$$

$$\Gamma^1_{11} = \frac{1}{2} \sum_{l=1}^{2} g^{1l} (\partial_1 g_{1l} + \partial_1 g_{1l} - \partial_l g_{11}) \qquad \partial_1 g_{11} = 0.$$

$$= \frac{1}{2} g^{11} (\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11})$$

$$= 0.$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22} \left(\cancelto{0}{\partial_1 g_{12}} + \cancelto{0}{\partial_1 g_{12}} - \partial_2 g_{11} \right) = -\frac{1}{2} g^{22} \partial_2 g_{11}$$

$$\partial_2 g_{11} = \partial_2 \left(f^2(u^2) \right) = 2ff'$$

$$\text{So } \Gamma^2_{11} = -\frac{1}{2} \frac{2ff'}{(f')^2 + (g')^2} = \frac{-ff'}{(f')^2 + (g')^2}.$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11} \left(\cancelto{0}{\partial_1 g_{21}} + \partial_2 g_{11} - \cancelto{0}{\partial_1 g_{12}} \right) = \frac{1}{2} g^{11} \partial_2 g_{11} = \frac{1}{2} \frac{1}{f^2} 2ff'$$

$$= \frac{f'}{f}.$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} \left(\cancelto{0}{\partial_1 g_{22}} + \cancelto{0}{\partial_2 g_{12}} - \cancelto{0}{\partial_2 g_{12}} \right) = 0.$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left(\cancelto{0}{\partial_2 g_{21}} + \cancelto{0}{\partial_2 g_{21}} - \cancelto{0}{\partial_1 g_{22}} \right) = 0$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) = \frac{1}{2} g^{22}\, \partial_2 g_{22}$$

$$\partial_2 g_{22} = \partial_2\left(f'(u^2)^2 + g'(u^2)^2\right) = 2f'f'' + 2g'g''.$$

$$\text{so } \Gamma^2_{22} = \frac{1}{2}\,\frac{1}{f'^2+g'^2}\left(2f'f'' + 2g'g''\right) = \frac{f'f''+g'g''}{(f')^2+(g')^2}$$

by Gauss Thm $\leadsto$ Egregium you can compute the Gauss curvature K.

Try this for other surfaces (e.g. sphere, hyperboloid, Enneper's surface, cylinder, torus, ...)